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1992 J. Phys. A: Math. Gen. 25 L1155

(http://iopscience.iop.org/0305-4470/25/19/004)

J. Phys. A: Math. Gen. 25 (1992) L1155–L1158. Printed in the UK

LETTER TO THE EDITOR

q-deformed paracommutation relations

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Received 3 June 1992

Abstract. The representations of an algebra of q-deformed paracommutation relations are discussed. The Bose quantization is shown to be exceptional among all possible para-Bose quantizations. Also it is demonstrated that the Ignatiev and Kuzmin oscillator is a particular case of the q-deformed paraoscillator.

During the past few years the theory of quantum groups and algebras [1, 2] has been intensively developed. The characteristic feature of these mathematical objects is an introduction of a deformation parameter q such that for $q=1$ the usual groups and algebras are restored. This activity has stimulated the study of the deformations of commutation relations for various annihilation and creation operators including Bose, Fermi, para-Bose and para-Fermi types [3–7]. In this letter we discuss the representations of the algebra of operators satisfying the q-deformed *paracommutation* relations. This algebra is interesting in that it is 'intermediate' between the para-Bose and para-Fermi algebras [8, 9].

The general aspects of paraquantization can be found in [9]. It is postulated that the annihilation and creation operators b and $b^\dagger$ obey the following trilinear commutation relations†:

$$[b,[b^\dagger,b]_\mp]=b \qquad [b^\dagger,[b^\dagger,b]_\mp]=-b^\dagger \tag{1}$$

where the upper (lower) sign corresponds to parafermions (parabosons). The representations of para-Fermi (para-Bose) algebra are (in)finite-dimensional. These representations are specified by negative (positive) half-integers which are the lowest energy levels for the Hamiltonian $H_0\equiv[b^\dagger,b]_\mp$.

The most natural way to build an object connecting para-Bose and para-Fermi oscillators is probably to deform the inner commutator. That is why we consider the trilinear commutation relations

$$[a,[a^\dagger,a]_q]=a \qquad [a^\dagger,[a^\dagger,a]_q]=-a^\dagger \tag{2}$$

where $a^\dagger=(a)^\dagger$, $[A,B]_1\equiv AB-qBA$ and $-1\leqslant q\leqslant 1$. Obviously the limiting cases $q=-1$ and $q=1$ correspond to the para-Bose and para-Fermi quantizations respectively. As in the paper [6], we postulate that there exists at least one representation in which the spectrum of the Hermitian operator (Hamiltonian) $N\equiv[a^\dagger,a]_q$ is bounded below. It follows from (2) that this spectrum has the form $N_n=N_0+n$, $n=0,1,\ldots$.

† Here and below we consider only one oscillator.

So, consider a Hilbert space with basis vectors $|n\rangle$ being the eigenvectors of N. In this representation the matrix elements of the annihilation and creation operators are [6]:

$$\langle n|a|m\rangle = \langle m|a^\dagger|n\rangle = \delta_{m,n+1}\sqrt{I(n)} \tag{3}$$

where

$$I(n) = -\sum_{m=0}^{n}\frac{N_0+m}{q^{m+1}}$$
$$= -\frac{1}{q^{n+1}(1-q)^2}[q^{n+2}(N_0+n) - q^{n+1}(N_0+n+1) - q(N_0-1) + N_0]. \tag{4}$$

The parameter N_0 determines the representation. Since the condition $I(0) > 0$ must be satisfied, it immediately follows that

$$N_0 q < 0. \tag{5}$$

Let, for some positive integer p, $I(p)$ be zero, that is, the representation is given by finite matrices of size $(p+1)\times(p+1)$. In this case N_0 is determined from (4) and depends on p via:

$$N_0 = -\frac{q}{1-q}\,\frac{pq^{p+1} - (p+1)q^p + 1}{1-q^{p+1}}. \tag{6}$$

Substituting this into (4), one gets for the matrix element squared

$$I^p(n) = \frac{1}{(1-q)(1-q^{p+1})}[(p-n)q^{p+1} - (p+1)q^{p-n} + n + 1]$$
$$= \frac{1}{(1-q^{p+1})}((p-n)[p+1;q] - (p+1)[p-n;q]) \tag{7}$$

where $n = 0, 1, \ldots, p-1$ and the notation $[n;q] \equiv (1-q^n)/(1-q)$ is used [10]. It is not difficult to show that there are no singularities for $q=1$ and the matrix elements coincide obviously with those for parafermionic creation and annihilation operators.

Since for given q the representations of the algebra (2) are specified by a value of N_0, it is convenient to depict the possible representations by curves on a plane (q, N_0) (figure 1). Due to the condition (5), these curves are placed in II and IV quadrants. From figure 1 one can see that the structure of the representations for positive and negative q is quite different. For arbitrary $q \in (0,1)$ there are finite-dimensional representations of *any* order $p \geq 1$, and N_0 is given by (6). In addition an infinite number of infinite-dimensional representations exist, for which $N_0 \leq -q/(1-q)$ (shaded region on figure 1). For arbitrary $q \in (-1, 0)$ there are finite-dimensional representations of any *even* order, i.e. when p is *odd*. However, for *even* p the situation changes. The finite-dimensional representations exist only for p greater than some p_0. This integer is determined from the inequality

$$q(p_0) < q < q(p_0-2) \tag{8}$$

where $q(p)$ is the root of the polynomial

$$\mathbb{P}(q,p) \equiv pq^{p+1} - (p+1)q^p + 1 \tag{9}$$

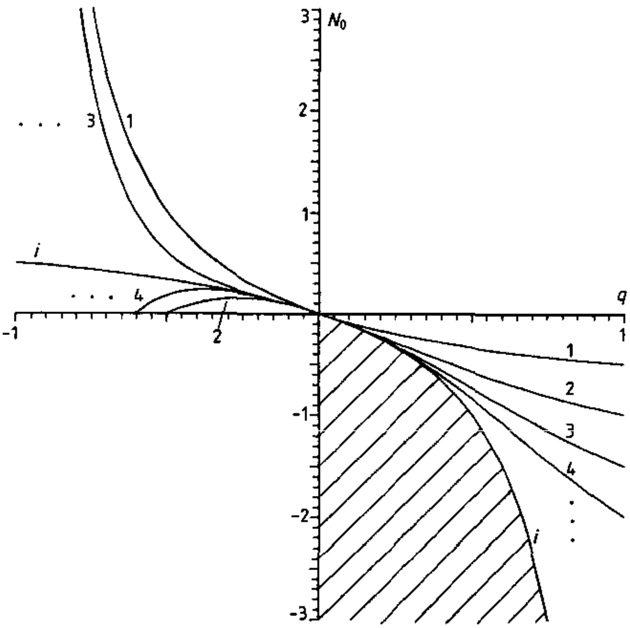

Figure 1. Each curve corresponds to a representation of the algebra (2). Shown are a few finite-dimensional representations only, the dimension of representation p being indicated by an integer. Curve $N_0 = -q/(1-q)$ is denoted by i. The shaded region corresponds to infinite-dimensional representations.

in the interval $(-1, 0)$. For even p the polynomial $\mathbb{P}(q, p)$ has only one simple root in this interval and $q(p) < q(p-2)$. We have not succeeded in deriving an explicit formula for $q(p)$ in the general case, so only the two biggest roots are given here:

$$\begin{aligned} q(2) &= -\tfrac{1}{2} \\ q(4) &= -\tfrac{1}{12}[(15(4\sqrt{6}+9))^{1/3} - (15(4\sqrt{6}-9))^{1/3}] - \tfrac{1}{4} \approx -0.606. \end{aligned} \tag{10}$$

Contrary to the case of positive q, *for* $q < 0$ *there is only one infinite-dimensional* representation which is determined by the following dependence of N_0 on q:

$$N_0 = -\frac{q}{1-q} \tag{11}$$

(this value of N_0 is obtained from (6) in the limit $p \to \infty$). For $q \to -1$ this representation becomes the canonical one for the Bose oscillator. Here we note a striking difference between limits $q \to 1$ and $q \to -1$. Taking the first limit one obtains all parafermionic representations, while in the second case only one of an infinite number of representations is obtained. Thus, the Bose quantization proves to be exceptional among all possibilities which correspond to the para-Bose quantization.

The value $q = 0$ is also peculiar. Although the formula (4) was derived in [6] excluding this q and is singular at first sight, equation (7) shows that there are no divergences for $q = 0$. For this q the matrix elements do not depend at all on the dimension of representation; are given by

$$I^{p}_{q=0}(n) = n + 1. \tag{12}$$

During the past few years the search for possible violation of the Pauli exclusion principle (PEP) has been intensively discussed [11-17]. This activity was initiated to a large extent by Ignatiev and Kuzmin (IK) [11] where the model of the non-usual fermionic oscillator was proposed. In this model two fermions are allowed to be in the same state with a small probability $\beta^2 \ll 1$ while the PEP strictly forbids it. It turns out that the annihilation (creation) operators $a_{\mathrm{IK}}(a^+_{\mathrm{IK}})$ introduced by IK are just the above described q-deformed operators represented by 3×3 matrices. More definitely, the relationship is the following:

$$a_{\mathrm{IK}} = \left(\frac{1+q+q^2}{q+2}\right)^{1/2} a \tag{13a}$$

$$\beta = \left(\frac{2q+1}{q+2}\right)^{1/2} \tag{13b}$$

where the matrix elements squared of a are given by equation (7) for $p=2$. It is seen that the limit $\beta \to 0$ when the IK oscillator becomes the ordinary fermionic one corresponds to $q \to q(2)$.

It is well known that the para-Fermi oscillator of order $(N-1)$ describes the N-level system [9]. The finite-dimensional representations of q-deformed paracommutation relations can be used to present the multi-level systems as well. In this approach the phenomenon of a decreasing number of levels appearing in the IK oscillator at $\beta \to 0$ has an analogue in multi-level systems. Consider the representation of algebra (2) by $N \times N$ matrices where $N = 2k+1$. If $q > q(N-1)$ then this representation corresponds to an N-level system. But for $q = q(N-1)$ the polynomial $\mathbb{P}(q, N-1)$ vanishes, and since $I^p(0) \propto \mathbb{P}(q, p)$, then $I^{N-1}(0)$ vanishes and the N-level system becomes the $(N-1)$-level one. In addition, for this value of q the equality $I^{N-1}(n) = I^{N-2}(n-1)$ holds.

Valuable discussions with Dr V V Semenov are cordially acknowledged.

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